THE RASMUSSEN INVARIANT OF ARBORESCENT AND OF MUTANT LINKS

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ABSTRACT. In the first section, an account of the definition and basic properties of the Rasmussen invariant for knots and links is given. An inequality involving the number of positive Seifert circles of a link diagram, originally due to Kawamura, is proven. In the second section, arborescent links are introduced as links arising from plumbing twisted bands along a tree. An algorithm to calculate their signature is given, and, using the inequality mentioned above, their Rasmussen invariant is computed. In the third section, an account of the concept of mutant links is given and an upper bound for the difference of the Rasmussen invariants of two mutant links is proven.

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Introduction

In the year 2000, Khovanov introducted a categorification of the Jones polynomial (see [16] for the original paper, [2] for an alternative account by Bar-Natan): a chain complex of graded abelian groups, the graded Euler characteristic of which equals the Jones polynomial.

This categorification, known as Khovanov homology, is strictly stronger than the Jones polynomial. On alternating knots, however, it is determined by the Jones polynomial together with the signature, as was conjectured in 2001 by Khovanov, Bar-Natan and Garoufalidis (see [2] and [10]).

The conjecture was proven by Lee in 2002 (see [17]) by introducing a variation of the Khovanov's chain complex, now called the Khovanov-Lee complex. This complex has quite a small homology; on knots, all information of that homology is encoded in the invariant s defined by Rasmussen in 2004 (see [20]). The Rasmussen invariant equals the signature on alternating knots and gives, like the signature, a lower bound for the slice genus. The first application of the Rasmussen invariant was thus to prove the Milnor conjecture about the slice genus of torus knots in a purely combinatorial way.

In 2005, Beliakova and Wehrli extended the definition of s to links (see [7]); on links, some of the nice properties of s are lost, but enough remain.

The first section of this thesis sums up all necessary definitions and statement from the different sources to define the Rasmussen invariant for links and prove its basic properties. To keep the first section as concise and purposeful as possible, while still providing the necessary tools for sections II and III, we do not elaborate on the connections between s and the Khovanov-homology; we manage to prove next to all results without introducing spectral sequences; and we do not introduce cobordisms, but merely analyze the effect of a fusion move on s. The first section is self-contained but for the proof of Reidemeister invariance of the homology of the Khovanov-Lee complex.

We construct an upper and a lower bound for the Rasmussen invariant and prove them directly from the definition. If the Seifert algorithm applied to a link diagram produces at most one Seifert circle adjacent to positive as well as negative crossings, then the lower bound equals the upper and hence determines the Rasmussen invariant. As an application of the inequality we give an alternative proof that the Rasmussen invariant equals the signature on non-split alternating links.

Arborescent links are a large class of links with interesting properties: they are fibered if and only if they arise from trees with weights equal to ± 2; it is known which of them are alternating; their genus is known; their crossing number can easily be computed from their tree; and they contain two-bridge knots, Pretzel knots, Montesinos knots and Slalom knots as subsets.

Their slice genus, however, is not known yet. In this thesis, their Rasmussen invariant and signature is computed; both invariants give a lower bound for the slice genus. To compute the Rasmussen invariant, we make use of the inequalities mentioned above, while the signature is computed directly from its definition as signature of the symmetric part of a Seifert matrix.

Mutation is an operation on links that was introduced by Conway. Different mutant knots are notoriously difficult to distinguish, numerous invariants like skein invariants, the colored Jones polynomial, signature or hyperbolic volume failing to do so. The three- and four-genus and quantum invariants of sufficient order are examples of invariants that in some cases distinguish pairs of mutant knots.

Khovanov homology is conjectured to be invariant under a link mutation that does not jumble different link components; this conjecture is partly proven (see III.2 for details). Given that the Rasmussen invariant is conjectured to be determined by Khovanov homology, it is worthwhile to analyze how the Rasmussen invariant changes under mutation. We are able to give an estimate.

The prerequisites of this thesis are a bit of linear algebra and some basic knot theory – definition of knots, links, diagrams, Reidemeister moves, Seifert surfaces, and the signature.

Finally, a word about enumeration of knots: there are two different systems to enumerate knots; the classical system follows the knot table published by Rolfsen [21], containing knots with crossing number up to ten. Rolfsen's table is correct but for the mistake of the Perko pair: two knots, 10_{161} and 10_{162} in the original notation, believed to be different turned out to be the same. So the knots were renumbered, and 10_{162} now refers to Rolfsen's 10_{163} a.s.o.

The other system is based on sorting knots lexicographically by their minimal Dowker-Thistlethwaite-notation; this is somewhat random, but has the advantage over the classical system of being well-defined for knots with arbitrarily high crossing number. Like KnotInfo [9] and Knot Atlas [5], we use the classical system for knots with crossing number 10 or less and the DT-system for knots with higher crossing number. Note that knotscape [13] uses the DT-system for all knots.

Sadly, there are incongruities concerning chiral knots: although in both enumeration systems it is theoretically clear to which mirror image of a knot a name refers, this is not always paid attention to. In this thesis, we try to be accurate.

I. The Rasmussen invariant

I.1. **Filtered vector spaces.** The Khovanov-Lee chain complex lives in the category of finite dimensional filtered vector spaces; so we begin with the necessary definitions and statements.

Throughout the thesis, all vector spaces are assumed to be over $\mathbb{Q}$.

List of definitions. A *finite dimensional filtered vector space* is a finite dimensional vector space V equipped with a *filtration*: an ascending chain of vector spaces

$$\{0\} = V_n \subset V_{n-1} \subset \ldots V_m = V$$

where the indices are integers. If $i > n$, we consider V_i to be $\{0\}$, and if $i < m$, we consider V_i to be V. Notice that the higher the index, the smaller the vector space; this is a matter of convention.

If a filtered vector space is denoted by $\square$, then by $\square_i$ we mean the i-th vector space of the chain of vector spaces constituting the filtration of $\square$, without explicitly saying so; unless, of course, $\square_i$ already denotes a different vector space.

We may define a filtration of V by just giving V_{i_j} for some i_j, where $i_1 < i_2 < \ldots < i_k$. In that case, if $i_j < k < i_{j+1}$, we mean $V_k = V_{i_{j+1}}$.

One says that $v \in V$ has *degree* k if $v \in V_k \setminus V_{k+1}$; this is denoted by $\deg_V v = k$. In case of unambiguity, the subscript indicating the vector space may be dropped.

Every element of V but the zero vector has a degree. If $k \in \mathbb{Z}$, let $V[k]$ be the filtered vector space $\{0\} = V[k]_{n+k} \subset \ldots \subset V[k]_{m+k} = V[k]$, where $V[k]_{i+k} = V_i$. This is called the *degree shift of* V *by* k.

A *filtered map* between filtered vector spaces V and W is an ordinary vector space homomorphism $f : V \to W$ that *respects the filtration*, i.e. $\forall i \in \mathbb{Z} : f(V_i) \subset W_i$. More generally, a *filtered map of degree* $k \in \mathbb{Z}$ is a a vector space homomorphism $f : V \to W$ satisfying $\forall i \in \mathbb{Z} : f(V_i) \subset W_{i+k}$. A map $f : V \to W$ is called *graded* if it is not only filtered, but also satisfies $\forall i \in \mathbb{Z} : f^{-1}(W_i) \subset V_i$.

Note that there is a crucial difference between vector space isomorphisms and isomorphisms of filtered vector spaces; by *isomorphism*, we will from now on mean the latter.

A *subspace* $U \subset V$ is a filtered vector space such that $\forall i \in \mathbb{Z} : U_i \subset V_i$. The inclusion map $U \hookrightarrow V$ is graded. If U is a subspace of V, we define the *quotient* V/U as the quotient in the category of vector spaces equipped with the filtration

$$(V/U)_i = \{[v] \in V \mid [v] \text{ has a representative in } V_i\}.$$

Notice that the canonical projection $V \twoheadrightarrow V/U$ is filtered.

The *filtered sum* of filtered vector spaces is given by summing pointwise, i.e. $(V \oplus W)_i = V_i \oplus W_i$. The *tensor product* is defined by

$$(V \otimes W)_i = \bigoplus_{j+k \geq i} V_j \otimes W_k.$$

A *filtered chain complex* $\mathcal{C}$ is a chain complex $\mathcal{C} = \ldots \to \mathcal{C}_i \xrightarrow{\partial_i} \mathcal{C}_{i+1} \to \ldots$ in the (abelian) category of filtered vector spaces and filtered maps; by some authors such an object would be called cochain complex, just as what is called Khovanov homology is a cohomology, strictly speaking. The *height shift by* $k \in \mathbb{Z}$ of $\mathcal{C}$ is denoted by $\mathcal{C}\{k\}$, where $\mathcal{C}\{k\}_i = \mathcal{C}\{k\}_{i+k}$, $\partial\{k\}_i = \partial_{i+k}$.

The *dual space* of V, denoted by V^*, is the space $\operatorname{Hom}(V, \mathbb{Q})$ with the filtration

$$w \in V^*_{-i} \quad :\Leftrightarrow \quad \forall v \in V_i : w(v) = 0.$$

Note that if V has elements of degrees $i_1, i_2, \ldots i_n$, then V^* has elements of degrees $-i_1 - 1, -i_2 - 1, \ldots - i_n - 1$. A basis $(v_1, \ldots v_n)$ of V induces a vector space isomorphism $\varphi : V \to V^*$ defined by $v_i \mapsto v_i^*$, where $v_i^*(v_j) = \delta_{ij}$. This map has the property $\deg_{V^*} \varphi(v) < -\deg_V v$ because $(\varphi(v))(v) \neq 0 \Rightarrow \varphi(v) \notin V^*_{-\deg_V v}$. The basis $(v_i^*, \ldots v_n^*)$ is called the *dual basis of* $(v_1, \ldots v_n)$.

The *dual of the map* $f : V \to W$, denoted by $f^* : W^* \to V^*$ is defined by $f^*(r)(v) = r(f(v))$. If f is filtered of degree k, then so is f^*. Note that $\dim \operatorname{im} f = \dim \operatorname{im} f^*$.

The *dual complex* of a given chain complex $\mathcal{C} = \ldots \to \mathcal{C}_i \xrightarrow{\partial_i} \mathcal{C}_{i+1} \to \ldots$ is the complex $\mathcal{C}^*$ with spaces $(\mathcal{C}^*)_i = \mathcal{C}_{-i}$ and boundary maps $(\partial^*)_i = (\partial_{-i-1})^*$.

In the remainder of I.1, we will prove a few technicalities.

Lemma I.1. Let V and $W' \subset W$ be filtered vector spaces. A filtered map $f : V \to W$ of degree k induces a filtered map $\widetilde{f} : V \to W/W'$ of degree k.

Proof. Let $v \in V$ have degree ℓ; then $f(v)$ is a representative of $[f(v)] \in W/W'$, and has degree at least $\ell + k$. So $[f(v)]$ has degree at least $\ell + k$, and hence $\widetilde{f}$ is filtered of degree k, too. $\square$

Corollary I.2. A filtered chain map of degree k descends to a map on homology that is filtered of degree k.

Lemma I.3. Let $\mathcal{C}$ be a chain complex of finite dimensional filtered vector spaces with homology H_i. Then the $(-i)$-th homology space of the dual complex $\mathcal{C}^*$ is isomorphic to H_i^*.

Proof. This boils down to the following: let $A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C$ be a sequence of filtered vector spaces and maps so that $g \circ f = 0$. Then the filtered vector spaces $\frac{\ker(f^*)}{\operatorname{im}(g^*)}$ and $\left(\frac{\ker g}{\operatorname{im} f}\right)$ are isomorphic. Let $\varphi : \frac{\ker(f^*)}{\operatorname{im}(g^*)} \to \left(\frac{\ker g}{\operatorname{im} f}\right)$ be defined by $\varphi([w])([b]) = w(b)$. We prove that φ is (i) well-defined, (ii) filtered, (iii) graded, (iv) injective and (v) bijective.

(i) Let $w \in \operatorname{im}(g^*)$, i.e. $\exists r \in C^* \; \forall b \in B : w(b) = r(g(b))$. Therefore, $\forall b \in \ker g : \varphi([w])([b]) = r(g(b)) = 0 \Rightarrow \varphi([w]) = 0$. On the other hand, let $b \in \operatorname{im} f$, i.e. $b = f(a)$ for an $a \in A$. Then $\varphi([w])([b]) = w(f(a)) = 0$ since $w \in \ker f^*$.

(ii) Let $[w] \in \left(\frac{\ker(f^*)}{\operatorname{im}(g^*)}\right)_k$. Without loss of generality, $w \in (\ker f^*)_k \subset B_k^*$. Let $[b] \in \left(\frac{\ker g}{\operatorname{im} f}\right)_{-k}$. Without loss of generality, $b \in (\ker g)_{-k} \subset B_{-k}$. Then $w \in B_k^*, b \in B_{-k} \Rightarrow \varphi([w])([b]) = w(b) = 0$. Therefore $\varphi([w]) \in \left(\frac{\ker g}{\operatorname{im} f}\right)_k^*$.

(iii) Let $\varphi([w]) \in \left(\frac{\ker g}{\operatorname{im} f}\right)_k^*$. By definition, this means $\forall [b] \in \left(\frac{\ker g}{\operatorname{im} f}\right)_{-k} : \varphi([w])([b])$ $= 0$, so $\forall b \in (\ker g)_{-k} : w(b) = 0$. Choose the filtered vector space B' so that B decomposes as the filtered sum $B' \oplus \ker g$. Let $w' \in B_k^*$ be defined by $w'|_{B'} = 0$ and $w'|_{\ker g} = w$. Then $(w - w')|_{\ker g} = 0$. As in the proof of (iv) below, one sees that this implies $(w - w') \in \operatorname{im} g^*$. Hence $[w] = [w'] \in \frac{\ker(f^*)}{\operatorname{im}(g^*)}$. Furthermore, because $B_{-k} = B'_{-k} \oplus (\ker g)_{-k}$, we have $w'|_{(\ker g)_{-k}} = 0 \Rightarrow w'|_{B_{-k}} = 0 \Rightarrow w' \in B_k^*$. Therefore, $[w] = [w'] \in \left(\frac{\ker(f^*)}{\operatorname{im}(g^*)}\right)_k$.

(iv) Let $\varphi([w]) = 0$, i.e. $w|_{\ker g} = 0$. Define $r \in C^*$ by $r(g(b)) = w(b)$; this is well-defined since $g(b) = g(b') \Rightarrow w(b) = w(b')$. Then $w(b) = g^*(r) \Rightarrow [w] = 0$.

(v) Note that

$$\begin{aligned}
\dim \ker(f^*)/\operatorname{im}(g^*) &= \dim \ker(f^*) - \dim \operatorname{im}(g^*) \\
&= \dim B^* - \dim \operatorname{im}(f^*) - \dim \operatorname{im}(g^*) \\
&= \dim B - \dim \operatorname{im} f - \dim \operatorname{im} g \\
&= \dim \ker g - \dim \operatorname{im} f \\
&= \dim(\ker g/\operatorname{im} f) \\
&= \dim(\ker g/\operatorname{im} f)^*
\end{aligned}$$

Because φ is injective and all vector spaces occurring are finite dimensional, φ is bijective. □

I.2. **The Khovanov-Lee complex.**

List of definitions. If D is a link diagram, we denote by $n(D)$ the number of crossings of D, by $n_\pm(D)$ the number of positive or negative crossings, respectively, and by $w(D) = n_+(D) - n_-(D)$ the writhe of D. If there is no danger of ambiguity, we just write $n, n_\pm$ and w instead of $n(D), n_\pm(D)$ and $w(D)$. Throughout I.2., let D be a diagram of an oriented link L.

Index the crossings of D by $1, \ldots n$. Each crossing has a 0- and a 1-resolution.

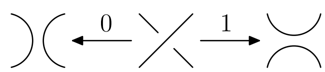


FIGURE 1. 0- and 1-resolution of a crossing

A simultaneous choice of resolution for all crossings is called a *state*; the set $S(D)$ of all states has 2^n elements. The *word* of state is the $0,1$-sequence of length n, where the i-th 0 or 1 stands for the choice of resolution of the i-th crossing. The height h of a state s with word $a_1 \ldots a_n$ is defined as the number of 1-resolutions of the state, i.e. $h(s) = \sum_{i=1}^n a_i$.

Resolving all crossings of D according to a state s yields a diagram, called diagram of the state, consisting only of circles, called the circles of the state. The set of these circles is denoted by $k(s)$. Let $V = \langle \mathbf{v}_-, \mathbf{v}_+ \rangle$ be a filtered, two-dimensional vector space, where $V_{-1} = V$ and $V_1 = \langle \mathbf{v}_+ \rangle$. Let $\mathbf{a} = \mathbf{v}_- + \mathbf{v}_+$ and $\mathbf{b} = \mathbf{v}_- - \mathbf{v}_+$.

To every state s we associate the filtered vector space

$$C_s = \Big(\bigotimes_{c \in k(s)} V_c \Big)[h(s) + w - n_-]$$

where $V_c = V$. Notice that the *pure tensors*, i.e. elements of C_s of the form $v = \bigotimes_{c \in k(s)} b_c$ generate C_s. We say v *colors the circle* c *with* b_c. In particular, we will often make use of the bases of pure tensors of $\mathbf{a}, \mathbf{b}$ or of $\mathbf{v}_\pm$.

An $\mathbf{a}$-colored circle and a $\mathbf{b}$-colored circle are said to be colored in the *opposite* way.

Let s and s' be two states that differ only in the resolution of the i-th crossing; such states are called *adjacent*. Let s be the state that 0-resolves the i-th crossing. Passing from the diagram of s to the diagram of s' may have one of two effects: if in the diagram of s the i-th crossing connects two different circles, these circles are merged to one circle; and if it connects one circle to itself, this circle is split in two.

We want to define a map $\partial_{s,s'} : C_s \to C_{s'}$ as tensor product of maps. For each circle c that is unaffected by the process of passing from s to s', take the identity map as factor; there is one additional factor: in case of a merge, a map $m : V \otimes V \to V$ and in case of a split a map $\Delta : V \to V \otimes V$, where we let

$$\begin{aligned} m(\mathbf{v}_+ \otimes \mathbf{v}_+) &= m(\mathbf{v}_- \otimes \mathbf{v}_-) = \mathbf{v}_+ & \Delta(\mathbf{v}_+) &= \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ \\ m(\mathbf{v}_+ \otimes \mathbf{v}_-) &= m(\mathbf{v}_- \otimes \mathbf{v}_+) = \mathbf{v}_- & \Delta(\mathbf{v}_-) &= \mathbf{v}_+ \otimes \mathbf{v}_+ + \mathbf{v}_- \otimes \mathbf{v}_-. \end{aligned}$$

Note that thanks to the degree shift in the definition of C_s the maps $\partial_{s,s'}$ are filtered.

Whenever the filtration is not of interest, it is easier to use the basis $(\mathbf{a}, \mathbf{b})$. With respect to that basis the maps m and Δ read:

$$\begin{aligned} m(\mathbf{a}\otimes\mathbf{a}) &= 2\mathbf{a} & \Delta(\mathbf{a}) &= \mathbf{a}\otimes\mathbf{a} \\ m(\mathbf{b}\otimes\mathbf{b}) &= -2\mathbf{b} & \Delta(\mathbf{b}) &= \mathbf{b}\otimes\mathbf{b} \\ m(\mathbf{a}\otimes\mathbf{b}) = m(\mathbf{b}\otimes\mathbf{a}) &= 0. \end{aligned}$$

If s and s' differ in the resolving of more than one crossing, we set $\partial_{s,s'} = 0$.

Remark I.4. The maps Δ and m may seem somewhat random; that is not the case: observe that the maps decompose as the sum of two graded maps, one of degree (-1) and the other of degree 3; taking the degree shift in consideration, these maps are graded maps of degree 0 and 4.

The 0-degree maps are up to scalar multiplication the only "non-trivial" symmetric maps; these are the maps defined by Khovanov in [16]. The 4-degree maps were added by Lee in [17] to show a certain identity of the Khovanov homology.

Definition. If D is a knot diagram, the writhe $w(D)$ is independent from orientation. That is not the case for links. So we define the *writhe of an orientation o of D* to be the writhe of D with orientation o.

Remark I.5. Let o, o' two orientations of L. Note that the Khovanov-Lee complex of (L, o) differs from the complex of (L, o') just by a degree shift of $\frac{3}{2}(w(o) - w(o'))$ and a height shift of $\frac{1}{2}(w(o) - w(o'))$.

Definition and Lemma I.6. For $i \in \mathbb{Z}$, let

$$\mathcal{C}_i = \bigoplus_{\substack{s\in S,\\ h(s)=i+n_-}} C_s, \quad \text{and} \quad \partial_i = \bigoplus_{\substack{s,s'\in S,\\ h(s)=i+n_-,\\ h(s')=i+n_-+1}} (-1)^{\mu(s,s')}\partial_{s,s'},$$

where $\mu(s, s')$ is the number of 1s occurring in the word of s prior to the digit where it differs from the word of s'. Then $\ldots \to \mathcal{C}_i \xrightarrow{\partial_i} \mathcal{C}_{i+1} \to \ldots$ is a chain complex called the *Khovanov-Lee complex of D*, or just the *complex of D*, and denoted by $\mathcal{C}(D)$.

We give a rigorous proof after having introduced two different ways to decompose the chain complex in spe as a sum of smaller complexes; until then, we treat $\mathcal{C}(D)$ as a mere sequence of spaces and maps.

Lemma I.7. All elements of the chain spaces have degree equal to $|L| \pmod 2$.

Proof. Let $s \in S(D)$ be a state and $\mathfrak{t} = \mathbf{v}_\pm \otimes \ldots \otimes \mathbf{v}_\pm \in \mathcal{C}_s$ be a pure tensor of $\mathbf{v}_\pm$. Let $\mathfrak{t}$ have k_+ many $\mathbf{v}_+$- and k_- many $\mathbf{v}_-$-factors. The degree of $\mathfrak{t} \in \mathcal{C}(s)$ equals $h(s) + w - n_- + k_+ - k_-$. Because $k_+ - k_- \equiv k_+ + k_- \equiv \#k(s) \pmod 2$, and $w - n_- \equiv n_+$, this equals $h(s) + n_+ + \#k(s)$ modulo two. Because $\mathcal{C}_s$ can be generated by pure tensors of $\mathbf{v}_\pm$, this is true of the degree of any element $\mathfrak{t} \in \mathcal{C}_s$.

Passing from a diagram D of L to the diagram of the Seifert state s_o can be seen as a process of resolving the crossings of L one after the other; each resolving changes the number of link components by one. We begin with a diagram that has $|L|$ link components and stop at a diagram that has $\#k(s_o)$: therefore $\#k(s_o) \equiv |L| + n \pmod 2$.

One can pass from any state $s \in S(D)$ to the Seifert state s_o by a sequence of fusions and merges. In each step, the number of cirles changes by one, as does

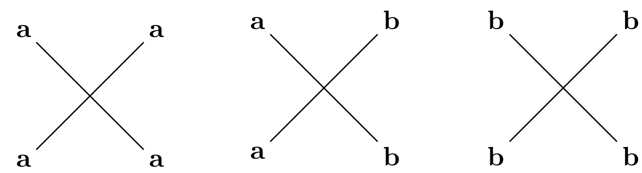


FIGURE 2. Possible local colorings of a crossing in an admissibly colored diagram.

the height of the state; thefore, $h(s) + \#k(s) \equiv h(s_o) + \#k(s_o)$. Consider that $h(s_o) = n_-$; so it follows that $h(s) + n_+ + \#k(s) \equiv \#k(s_o) + n \equiv |L| \pmod 2$. □

Remark I.8. We have already remarked that the boundary maps are the sum of a map that preserves the degree and one map that increases the degree by four. So $\mathcal{C}(D)$ is the filtered sum of two complexes: one complex containing all elements of degree equal to $|L| \pmod 4$, the other complex containing all elements equal to $|L| + 2 \pmod 4$. Likewise, homology – once we have proven $\mathcal{C}(D)$ is a chain complex – is the filtered sum of the two homologies, one summand containing all elements of degree equal to $|L| \pmod 4$, the other all elements of degree equal to $|L| + 2 \pmod 4$.

Definition. A space homeomorphic to the closed unit interval I is called an *arc*. An *edge* of a diagram is either a circle that contains no crossings, or an arc that has two crossings as endpoints and contains no crossings in its interior.

Remark I.9. Let D be a diagram of L. A *coloring* of D is a map that assigns to each edge of the diagram either **a** or **b**. A coloring is called *admissible* if there is a way to resolve all crossings so that each of the emerging circles is of one color only (see figure 2).

Now every chain space of the Khovanov-Lee complex can be generated by pure tensors of $\mathbf{a}, \mathbf{b}$. Each of these generators belongs to an admissible coloring: color every edge by **a** (or **b**) if it belongs to an **a**- (or **b**)-circle in the diagram of the state in which the generator in question lives. Conversely, to each admissible coloring c we assign a subset $V(c)$ of these generators. The crucial point is that a generator $\mathfrak{t} \in V(c)$ is mapped by the boundary map to a non-zero multiple of another generator $\mathfrak{t}' \in V(c)$, as one can see from the definition of m and Δ. Hence $V(c)$ for every c is a subcomplex and $\mathcal{C}(D)$ is the direct sum of these subcomplexes.

Notice that this decomposition does not respect the degree; and note that this remark does not presume that $\mathcal{C}(D)$ is indeed a chain complex.

Proof of lemma I.6. Because of the above remark it is sufficient to prove that if c is an admissible coloring, then $V(c)$ is a chain complex. It is enough to prove that if s_1, s_2, s_2', s_3 are states and s_1 and s_3 are adjacent to s_2 and s_2', and $V(c)$ is supported in these states, then

$$(-1)^{\mu(s_1,s_2)+\mu(s_2,s_3)}\partial_{s_2,s_3} \circ \partial_{s_1,s_2} + (-1)^{\mu(s_1,s_2')+\mu(s_2',s_3)}\partial_{s_2',s_3} \circ \partial_{s_1,s_2'} = 0$$

We show that

$$\partial_{s_2,s_3} \circ \partial_{s_1,s_2} = \partial_{s_2',s_3} \circ \partial_{s_1,s_2'}$$

and

$$(-1)^{\mu(s_1,s_2)+\mu(s_2,s_3)} = -(-1)^{\mu(s_1,s_2')+\mu(s_2',s_3)}.$$

The chain spaces are generated by pure tensors of **a**, **b**. The functions $\partial_{s_2,s_3} \circ \partial_{s_1,s_2}$ and $\partial_{s_2',s_3} \circ \partial_{s_1,s_2'}$ map a generator $\mathfrak{s}$ in $V(c) \cap C_{s_1}$ to a non-zero multiple of a generator in $V(c) \cap C_{s_3}$. Because $V(c) \cap C_{s_3}$ is one-dimensional, both functions map $\mathfrak{s}$ to a non-zero multiple of the same generator $\mathfrak{t}$. So it just remains to prove that the rational coefficient are the same. Notice that passing from s_1 to s_3 via s_2 or s_2' entails the same number of times of splitting a circle, merging two **a**-circles and merging two **b**-circles. Therefore, considering the coefficients of m and Δ, the image of $\mathfrak{s}$ has the same rational coefficient, whether one passes from s_1 to s_3 via s_2 or s_2'. Thus $\partial_{s_2,s_3} \circ \partial_{s_1,s_2} = \partial_{s_2',s_3} \circ \partial_{s_1,s_2'}$.

There are two crossings c and c' such that s_1 0-resolves both, s_3 1-resolves both, s_2 0-resolves c and 1-resolves c', and s_2' 0-resolves c' and 1-resolves c. Without loss of generality, let c have a higher index than c'. Then $\mu(s_2, s_3) = \mu(s_1, s_2') + 1$ and $\mu(s_1, s_2) = \mu(s_2', s_3)$, and therefore

$$(-1)^{\mu(s_1,s_2)} \cdot (-1)^{\mu(s_2,s_3)} = -(-1)^{\mu(s_1,s_2')} \cdot (-1)^{\mu(s_2',s_3)}.$$

□

Definition and Lemma I.10. Let L be a link. By $H_i(L)$ we denote the i-th filtered homology space of the Khovanov-Lee chain complex of the above lemma. Let the *homology of* L be the sum of all homology spaces and denoted by $H^\star = \bigoplus_{i\in\mathbb{Z}} H_i$.

This definition is justified because the isomorphism type of H as filtered vector space is a link invariant.

Rasmussen proves this in section 6 of [20] by constructing chain maps between chain complexes of diagrams related by the Reidemeister moves; such that these chain maps induce vector space isomorphisms on homology. For this, the degree shift of $w-n_-$ and the height shift of $-n_-$ are relevant, similar to the term $(-A)^{-3w}$ in the definition of the Jones polynomial. Then an argument involving spectral sequences shows that the filtration of homology is not changed under Reidemeister moves, either.

Remark I.11. The Khovanov complex is constructed using $\mathbb{Z}$- instead of $\mathbb{Q}$-modules. The difference is that $\mathbb{Z}$-modules, i.e. abelian groups, allow torsion; but torsion is not of interest from the viewpoint of the Rasmussen invariant, so we can work over $\mathbb{Q}$ with clear conscience.

I.3. **Cobordisms and definition of the Rasmussen invariant s for links.** It turns out the dimension of the homology is just $2^{|L|}$ and generators can explicitly be given.

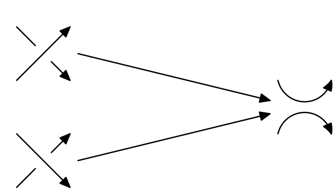


FIGURE 3. Seifert resolution of a crossing

Definition. Let o be an orientation of L. An orientation induces a state s_o that resolves every crossing in harmony with the orientation, as in the Seifert algorithm (see figure 3). This state is called *Seifert resolution of the orientation*. Its circles are called *Seifert circles*. Now color every circle of this state that is oriented counterclockwise with **a** and every circle oriented clockwise with **b**. Switch the coloring for all circles that are contained in an odd number of other circles, where a circle is said to be contained in another circle if it lies in the compact one of the two connected components of the complement of that other circle. The element of $C(s_o)$ that colors the circles like this is called the *canonical generator corresponding to o* and is denoted by $\mathfrak{s}_o$.

Remark I.12. Note that $h(s_o) = n_-$, so $C_{s_o} = \left(\bigotimes_{c \in k(s_o)} V_c\right)[w]$. Furthermore, $\mathfrak{s}_o \in C_0$.

Lemma I.13. In the diagram of s_o each crossing connects two circles. One of them is colored with **a** by $\mathfrak{s}_o$, and the other one with **b**.

Proof. If two circles are connected by a crossing, they locally look like this:)(.

There are two cases: either neither circle contains the other; then one of the circles is oriented clockwise and the other one counterclockwise. Or one of the circles does contain the other; then they have the same orientation.

In both cases, they are colored oppositely. □

Lemma I.14. The homology classes of the $2^{|L|}$ canonical generators are a basis of homology.

Proof. Decompose $\mathcal{C}(D)$ as a sum of subcomplexes as in remark I.9. Then $H^\star(D)$ is the direct sum of the homologies of these subcomplexes. Let c be an admissible coloring that contains a crossing at which all four strands are colored the same (called coloring of type I). Then, $V(c) = V_0(c) \sqcup V_1(c)$, where $V_0(c)$ contains the elements of the states that 0-resolve that crossing, and $V_1(c)$ contains the elements of the states that 1-resolve that crossing. The boundary map is an isomorphism from $V_0(c)$ to $V_1(c)$. Hence the homology of $V(c)$ vanishes.

On the other hand, let c be an admissible coloring that uses both colors at each crossing (called coloring of type II). Then there is only one way to resolve each crossing consistently with the coloring, and therefore only one state that contains elements of $V(c)$. So $V(c)$ is one-dimensional and isomorphic to its homology.

Now the set of colorings of type II is in 1-1-correspondence with the set of possible orientations of D: If c is a coloring c of type II, it is supported in one state only. In that state, orient every circle colored **a** counterclockwise and every circle colored **b** clockwise. Reverse the orientation of every circle that is contained in an odd number of other circles. This orientation of circles induces an orientation of the diagram D.

On the other hand, if an orientation o is given, resolve D in Seifert's way and color each circle as in the above definition of canonical generators. This induces an admissible coloring; that it is of type II is precisely the statement of lemma I.13 above. It is obvious that the described correspondence between type II colorings and orientations is 1-1. Notice that $V(c)$ is generated by $\mathfrak{s}_o$. So the canonical generators generate homology. □

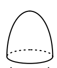
Remark I.15. This proof comes from Wehrli (see [27]). Lee's original proof (see [17]) relies on induction on the number of link components and the number of crossings and uses a long exact sequence in Khovanov-Lee homology.

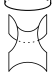
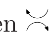
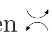
Definition. Let $L_0, L_1 \in S^3$ be links. A *cobordism* from L_0 to L_1 is a smooth, oriented surface S properly embedded[1] in $[0,1] \times S^3$ so that $S \cap \{i\} \times S^3 = L_i$ for $i = 1, 2$.

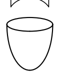
Remark I.16. A weakly connected[2] cobordism induces an automorphisms of vector spaces from $H(L_0)$ to $H(L_1)$. If one follows the construction of the chain complex given by Bar-Natan in [3], this comes naturally. With our construction, one has to decompose the cobordism S as sum of elementary cobordisms first, as done by Rasmussen [20, section 4].

It is possible to fix a projection $S^3 \to \mathbb{R}^2$ so that the set of $i \in [0,1]$ for which the projection of $S \cap (\{i\} \times S^3)$ is not a link diagram is discrete. The cobordism S can then be depicted as a sequence $D_0, \dots D_n$ of link diagrams so that:

- Each D_j is the projection of $S \cap (\{i_j\} \times S^3)$, where $0 = i_0 < i_1 < \dots < i_n = 1$.
- There is at most one k in between $i_j < k < i_{j+1}$ such that $S \cap (\{k\} \times S^3)$ is not a link diagram. For ℓ with $i_j \leq \ell < k$, the projection $S \cap (\{\ell\} \times S^3)$ is related to D_{i_j} by an isotopy of the plane; for ℓ with $k < \ell \leq i_{j+1}$, the projection $S \cap \{\ell\} \times S^3$ is related to $D_{i_{j+1}}$ by an isotopy of the plane.
- One can pass from D_{i_j} to $D_{i_{j+1}}$ by either a Reidemeister move or one of the three *Morse moves*:

 *birth* – add a circle to the diagram; comes from a cap-cobordism.

 *fusion* – either switching between  and , or between  and ; comes from a saddle cobordism.

 *death* – remove a circle from the diagram; comes from a cup-cobordism.

We can define the map $H(L_0) \to H(L_1)$ for each of the six elementary cobordisms – three Reidemeister moves and three Morse moves – and for the sum of elementary cobordisms by composition. It turns out that $|s(L_0) - s(L_1)| \leq \chi(S)$.

However, the only application needed in this thesis is the effect of a fusion on the Rasmussen invariant; this can be analyzed without even defining cobordisms, so we will not make the above digressions more formal.

Lemma I.17. Let D_1 be a diagram of the link L_1 and D_2 a diagram of the link L_2 obtained from D_1 by a fusion. Let o be an orientation of D_1; it induces an orientation o' of D_2. Then there is a filtered map of degree -1 from $H_i(L_1) \to H_i(L_2)$ that takes a canonical generator $\mathfrak{s}_o$ to a non-zero multiple of the canonical generator $\mathfrak{s}'_{o'}$.

Proof. There is an obvious one-to-one correspondence between the states of D_1 and D_2, denote it by $\varphi : S(D_1) \to S(D_2)$. Furthermore, the diagrams of $s \in S(D_1)$ and $\varphi(s) \in S(D_2)$ look the same except for the site where the fusion took place: thus,

[1]In these circumstances properness just means that the image of the boundary of S is the intersection of the image of S and $\{0,1\} \times S^3$.

[2]A cobordism is *weakly connected* if the intersection of each of its connected components with L_0 is non-empty.

the diagram of $\varphi(s)$ has either one more or one less circle than the diagram of s, depending on whether the fusion splits or merges two circles. So we can define a map $\partial_{s,\varphi(s)}$ in the usual way. This map is filtered of degree -1.

One can show that $\partial(s', \varphi(s')) \circ \partial(s, s') = \partial(\varphi(s), \varphi(s')) \circ \partial(s, \varphi(s))$ in the same way we proved that the Khovanov-Lee complex is a chain complex (see proof of I.6).

Let f_i be the sum of $\partial_{s,s'}$ for all $s \in S(D_1)$ and $s' \in S(D_2)$ with $h(s) = h(s') = i$. Then it follows that $f_{i+1} \circ \partial_i = \partial_i \circ f_i$, so the maps f_i constitute a chain map f from $\mathcal{C}(L_1)$ to $\mathcal{C}(L_2)$. The map f is filtered of degree -1 and induces a map of degree -1 from $H^*(L_0) \to H^*(L_1)$ (see corollary I.2).

Let $s_o \in S(D_1)$ and $s'_o \in S(D_2)$ be the Seifert states. Let $\mathfrak{s}_o \in C_{s_o}$ and $\mathfrak{s}'_o \in C_{s'_o}$ be the canonical generators. Consider the diagrams of s_o and s'_o: near the fusion site, one of them has one and the other one two circles. But these three circles are colored the same by $\mathfrak{s}_o$ and $\mathfrak{s}'_o$. Hence, up to non-zero scalar multiplication, f sends $\mathfrak{s}_o$ to $\mathfrak{s}'_o$. □

Lemma I.18. Let L be a link with orientation o. Then the subspace of homology generated by $\mathfrak{s}_o$ and $\mathfrak{s}_{\overline{o}}$ has elements of two different degrees, the difference of which is two. Both canonical generators have the lower degree; the higher degree is attained by either $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$ or $\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}$.

Proof. Since the subspace in question is generated by two elements, it could be supported in one or in two different degrees. As seen in remark I.8, the chain complex of L decomposes as direct sum of two chain complexes. We show that $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$ and $\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}$ lie in one of those two chain complexes each. It follows that the homological degree of $[\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}]$ and $[\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}]$ differs by 2 (mod 4).

Consider $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$. It can be written as sum of elements of the form $\mathbf{v}_\pm \otimes \ldots \otimes \mathbf{v}_\pm$ in a unique way. In that sum, $\mathbf{v}_\pm \otimes \ldots \otimes \mathbf{v}_\pm$ appears as summand with non-zero coefficient if and only if the number of $\mathbf{v}_+$-factors is even. All pure tensors of $\mathbf{v}_\pm$ with an even number of $\mathbf{v}_+$-factors form a basis of one of the two chain complexes mentioned above. Similarly, $\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}$ is the sum of all elements $\mathbf{v}_\pm \otimes \ldots \otimes \mathbf{v}_\pm$ that have an odd number of $\mathbf{v}_+$-factors and consequently lies in the other one of the two chain complexes mentioned above.

It remains to prove that the difference of the two gradings indeed equals 2, not only mod 4. Pick any circle. Apply a fusion move that splits this circle, then merge again with another fusion move. The composition of the maps associated to these two moves (see lemma I.17) send $\mathbf{a} \mapsto 2\mathbf{a}$ and $\mathbf{b} \mapsto -2\mathbf{b}$. Because one of the canonical generators $\mathfrak{s}_o$ and $\mathfrak{s}_{\overline{o}}$ colors that circle with $\mathbf{a}$, and the other one with $\mathbf{b}$, the elements $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$ and $\pm 2(\mathfrak{s}_o - \mathfrak{s}_{\overline{o}})$ are interchanged (up to non-scalar multiplication). Thus, by lemma I.17, the homological degree of $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$ and of $\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}$ differ by at most 2; since we already know that the degree differs by 2 (mod 4), the difference is 2.

Finally, the sum of two elements of different degree has the lower of the two degrees: so

$$\mathfrak{s}_\mathfrak{o} = \frac{(\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}) + (\mathfrak{s}_o - \mathfrak{s}_{\overline{o}})}{2}$$

– and similarly $\mathfrak{s}_{\overline{\mathfrak{o}}}$ – has the lower of the two degrees. □

Definition. The *Rasmussen invariant* s of the oriented link L is defined as the number between the two integers of difference 2 that arise as degrees of elements of the subspace of homology that is generated by $\mathfrak{s}_o$ and $\mathfrak{s}_{\overline{o}}$, where o is the orientation of L. Another way to say that is

$$s = \frac{\deg_{H^\star}(\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}) + \deg_{H^\star}(\mathfrak{s}_o - \mathfrak{s}_{\overline{o}})}{2}.$$

If o is an orientation of L, by $s(L, o)$ we denote the Rasmussen invariant of L oriented by o.

I.4. **Basic properties of s.**

Lemma I.19. Let L be a link. Then

$$s(L) \equiv |L| + 1 \pmod 2$$

Proof. The Rasmussen invariant s of L is equal to one plus the degree of some element of C_0; by lemma I.7 the degree of such an element is equal to $|L| \pmod 2$. □

Lemma I.20. Let D_1 be a diagram of the link L_1 and D_2 a diagram of the link L_2 obtained from D_1 by a fusion: switching between $\asymp$ and $)($, or between $\asymp$ and $)($. Then

$$s(L_2) = s(L_2) \pm 1$$

Proof. Let f be the filtered map of lemma I.17. This map f is of degree -1, so $\deg_{H^\star(L_0)} \mathfrak{s}_o - 1 \geq \deg_{H^\star(L_L)} \mathfrak{s}'_o \Leftrightarrow s(L_0) - 1 \geq s(L_1)$. Switching the role of L_0 and L_1 results in $s(L_1) - 1 \geq s(L_0) \Rightarrow |s(L_0) - s(L_1)| \leq 1$.

Since the number of components of L_0 and L_1 differs by one, so does their Rasmussen invariant mod 2 (see lemma I.19). This excludes the case and thus completes the proof. □

Lemma I.21. Let L be a link. Then the Khovanov-Lee complex of $\overline{L}$ is isomorphic to the dual of the complex of L with a degree shift of one:

$$\mathcal{C}(\overline{L}) \cong \mathcal{C}(L)^*[1].$$

Proof. We (i) construct an isomorphism $\mathcal{C}_i(\overline{L}) \to \mathcal{C}_{-i}(L)$, (ii) construct an isomorphism $\mathcal{C}_{-i}(L) \to (\mathcal{C}_i(L))^*[1]$, and (iii) prove that the composition $\psi_i : \mathcal{C}_i(\overline{L}) \to (\mathcal{C}(L)^*)_i[1]$ of the two isomorphisms is a chain map.

(i) Let D be diagram of L and $s \in S(D)$ a state of D. Let $\overline{s} \in s(\overline{D})$ be the state that resolves every crossing the same way s does. Then there is natural isomorphism between $\mathcal{C}(L)_s$ and $\mathcal{C}(\overline{L})_{\overline{s}}$. A crossing is 0-resolved by $\overline{s}$ if it is 1-resolved by s, and vice versa. So $h(\overline{s}) = n - h(s)$. The space $\mathcal{C}(L)_s$ is a summand of $\mathcal{C}(L)_{h(s)-n_-}$, whereas the space $\mathcal{C}(\overline{L})_{\overline{s}}$ is a summand of $\mathcal{C}(\overline{L})_{h(\overline{s})-n_+} = \mathcal{C}(\overline{L})_{n_- - h(s)}$, because the number of negative crossings of $\overline{D}$ is n_+. So there are natural isomorphisms between $\mathcal{C}(L)_i$ and $\mathcal{C}(\overline{L})_{-i}$.

(ii) We map $V \to V^*[1]$ by $\mathbf{v}_\pm \mapsto \mathbf{v}^*_\mp$. This is an isomorphism. It extends to an isomorphism $V \otimes \ldots \otimes V \to (V \otimes \ldots \otimes V)^*[1]$, so we have isomorphisms from $\mathcal{C}_s(L) \to (\mathcal{C}_s(L))^*[1]$, where $s \in S(D)$. These isomorphisms in turn extend to isomorphisms $\mathcal{C}_{-i}(L) \to (\mathcal{C}_i(L))^*[1]$.

(iii) Let $s, s' \in S(\overline{D})$ be adjacent states so that $\partial_{s,s'}$ is non-zero. Assume $\partial_{s,s'} : C_s(\overline{D}) \to C_{s'}(\overline{D})$ is a merge (if it is a split, the argument is similar). Then the map $\partial_{s',s} : C_{s'}(D) \to C_s(D)$ is a split. So in the dual complex, the crucial part of the map $\partial^*_{s',s} : C_s(D)^* \to C_{s'}(D)^*$ is Δ^*. But the isomorphism $\mathbf{v}_\pm \mapsto \mathbf{v}^*_\mp$ maps m to Δ^*, therefore $\partial^*_{s',s} \circ \psi_{h(s')} = \psi_{h(s)} \circ \partial_{s,s'}$. □

Corollary I.22. If L is a link, then for all $i \in \mathbb{Z}$ the i-th homology space $H_i(\overline{L})$ is isomorphic to $(H_i(L))^*[1]$.

Proof. This follows from the previous lemma I.21 and the fact that the dual chain complex has dual homology (see lemma I.3). □

Proposition I.23. Let L and L' be links.

(i) $s(L_1 \sqcup L_2) = s(L_1) + s(L_2) - 1$.

(ii) $2 - 2|L| \leq s(L) + s(\overline{L}) \leq 0$. Let o be the orientation of L. The only values $s(\overline{L})$ can take are $-s(L, o')$, where o' is an orientation of L with same writhe as o, or $-s(L, o') + 2$, if there is another orientation o'' of L with same writhe as o that satisfies $s(L, o') - 4 = s(L, o'')$.

(iii) $s(L_1 \# L_2) = s(L_1) + s(L_2) - 1 \pm 1$. If $s(\overline{L_i}) = -s(L_i)$ for $i \in \{1, 2\}$ and $s(\overline{L_1 \# L_2}) = -s(L_1 \# L_2)$, which is e.g. the case for knots, then $s(L_1 \# L_2) = s(L_1) + s(L_2)$.

Proof. **(i)** Let D, D_1 and D_2 be diagrams of L, L_1 and L_2, respectively. There is a natural bijection $\varphi : S(D_1) \times S(D_2) \to S(D_1 \sqcup D_2)$. We have $h(s_1) + h(s_2) = h(\varphi(s_1, s_2))$ and $n_-(D) = n_-(D_1) + n_-(D_2)$. Thefore

$$\mathcal{C}(D)_i = \bigoplus_{h(s)=i+n_-(D)} \mathcal{C}(D)_s$$

$$= \bigoplus_{j+k=i} \Big(\bigoplus_{h(s_1)=j+n_-(D_1)} \mathcal{C}(D_1)_{s_1} \Big) \otimes \Big(\bigoplus_{h(s_2)=k+n_-(D_2)} \mathcal{C}(D_2)_{s_2} \Big) = \bigoplus_{j+k=i} \mathcal{C}(D_1)_j \otimes \mathcal{C}(D_2)_k$$

So we have $\mathcal{C}(L_1 \sqcup L_2) = \mathcal{C}(L_1) \otimes \mathcal{C}(L_2) \Rightarrow H^\star(L_1 \sqcup L_2) = H^\star(L_1) \otimes H^\star(L_2)$.

The canonical generator of $L_1 \sqcup L_2$ is the tensor product of the canonical generators of L_1 and L_2, so its homological degree is the sum of the homological degrees of the canonical generators of L_1 and L_2. This proves the claim.

(ii) To prove the upper bound: by the above corollary I.22, $H_i(\overline{L}) \cong (H_i(L))[1]^*$. The basis of pure tensors of $\mathbf{v}_\pm$ induces a vector space isomorphism $\varphi : H_i(L) \to (H_{-i}(L))[1]^*$ with the property $\deg \varphi(\mathfrak{t}) \leq -\deg \mathfrak{t}$. So there is a vector space isomorphism $\widetilde{\varphi} : H_i(L) \to H_{-i}(\overline{L})$ with the same property. We denote the canonical generators of L by $\mathfrak{s}_o$ and those of $\overline{L}$ by $\bar{\mathfrak{s}}_o$. Choose $j \in \{0, 1\}$ so that $\deg_{H^\star(L)} \mathfrak{s}_o + (-1)^j \mathfrak{s}_{\overline{o}} = s(L) + 1$. The function $\widetilde{\varphi}$ maps $\mathfrak{s}_o + (-1)^j \mathfrak{s}_{\overline{o}}$ to $\bar{\mathfrak{s}}_o + (-1)^j \bar{\mathfrak{s}}_{\overline{o}}$. Therefore we have

$$\begin{aligned} \deg_{H^\star(\overline{L})} \bar{\mathfrak{s}}_o + (-1)^j \bar{\mathfrak{s}}_{\overline{o}} &\leq -\deg_{H^\star(L)} \mathfrak{s}_o + (-1)^j \mathfrak{s}_{\overline{o}} \Rightarrow \\ s(\overline{L}) \pm 1 &\leq -(s(L) + 1) \Rightarrow \\ s(\overline{L}) + s(L) &\leq -1 \pm 1 \Rightarrow \\ s(\overline{L}) + s(L) &\leq 0. \end{aligned}$$

To prove the lower bound: it is possible to transform $L \sqcup \overline{L}$ to the trivial link with $|L|$ components by $|L|$ fusion moves. Therefore

$$\begin{aligned} |s(L \sqcup \overline{L}) - (1 - |L|)| &\leq |L| \Rightarrow \\ |s(L) + s(\overline{L}) - 2 + |L|| &\leq |L| \Rightarrow \\ -2 - 2|L| &\leq s(L) + s(\overline{L}). \end{aligned}$$

Finally, consider again that $H_i(\overline{L})$ and $(H_i(L))^*[1]$ are isomorphic. So $H_i(\overline{L})$ has an element of degree k if and only if $H_{-i}(L)$ has an element of degree $-k$; this is the case if and only if there is an orientation o' of L so that $\mathfrak{s}_{o'}$ lives in H_i and $k = s(L,o') \pm 1$. Furthermore, $\mathfrak{s}_{o'}$ lives in H_i if and only if o' has the same writhe as o. So, $\mathfrak{s}_o$ is mapped to an element of degree $-s(L,o') - 1$ or $-s(L,o') + 1$. In the first case, $s(\overline{L}) = -s(L,o')$. In the second, $s(\overline{L}) = -s(L,o') + 2$; but since there must be an element of $H_i(\overline{L})$ with degree two greater than the image of $\mathfrak{s}_o$, this may only happen if there is an orientation o'' of L with same writhe as o such that $s(L,o'') = s(L,o') - 4$ or $s(L,o'') = s(L,o') - 2$. The first case is part of the claim of the lemma, and in the second case $s(\overline{L}) = -s(L,o')$.

(iii) There is an obvious fusion move relating $L_1 \# L_2$ to $L_1 \sqcup L_2$. Therefore, by lemma I.20, $s(L_1 \# L_2) = s(L_1 \sqcup L_2) \pm 1 = s(L_1) + s(L_2) - 1 \pm 1$.

If $s(\overline{L_i}) = -s(L_i)$ for $i \in \{1,2\}$ and $s(\overline{L_1 \# L_2}) = -s(L_1 \# L_2)$, then

$$\begin{aligned} s(L_1 \# L_2) &= s(\overline{\overline{L_1} \# \overline{L_2}}) \\ &= -s(\overline{L_1} \# \overline{L_2}) \\ &= -(s(\overline{L_1}) + s(\overline{L_2}) - 1 \pm 1) \\ &= -(-s(L_1) - s(L_2) - 1 \pm 1) \\ &= s(L_1) + s(L_2) + 1 \pm 1 \end{aligned}$$

Therefore $s(L_1 \# L_2) = s(L_1) + s(L_2)$. □

Remark I.24. There are currently no links L_1, L_2 known for which $s(L_1 \# L_2) = s(L_1) + s(L_2) - 2$ ([28]). So proving that there are indeed none appears to be an open problem. This would be an easy corollary of invariance of the Rasmussen invariant under link mutation (see section III.2): Let U be the unknot, then $U \sqcup (L_1 \# L_2)$ and $L_1 \sqcup L_2$ are mutant, so $s(L_1 \# L_2) = 1 + s(U \sqcup (L_1 \# L_2)) = 1 + s(L_1 \sqcup L_2) = s(L_1) + s(L_2)$.

The estimate for $s(\overline{L})$, however, is sharp in the sense that both lower and upper bound are attained: For example, the n-component trivial link U_n satisfies $s(U_n) = 1 - n$, and so does its mirror image. There are even non-split links L that do not satisfy $s(\overline{L}) = -s(L)$: let L be a n-component link with unknotting number u. A crossing change may be realized as two fusion moves (see figure 4). So there is a

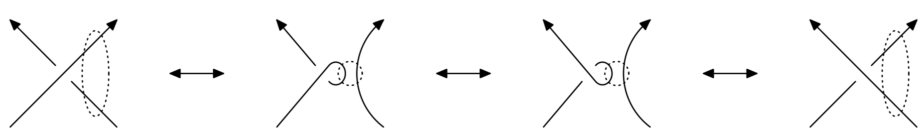


FIGURE 4. A crossing change realized as two fusion moves

sequence of $2u$ fusion moves from L to U_n, and therefore $s(L) \leq 1 - n + 2u$. This inequality holds for the mirror image as well. So for any link $1 - n + 2u < 0 \Rightarrow s(\overline{L}) \neq -s(L)$. Take e.g. the Whitehead link and replace one component with its $(n-1)$-cable (see figure 5 for the case $n = 4$). The resulting link has unknotting number 1, so for $n \geq 4$ we have $1 - n + 2u < 0$. This was Stephan Wehrli's idea (see [28]).

More generally, take a link L with $|L| \geq 2$ that becomes trivial if one removes a certain link component from it, e.g. any Brunnian link (taking the Hopf link will

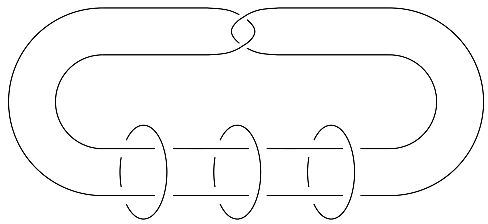


FIGURE 5. A non-split link L with $s(\overline{L}) \neq -s(L)$.

lead to the above example). Replacing this component by its Whitehead double yields a link with unknotting number 1. Now, by replacing any other component with its n-cable one can increase the number of link components without changing the unknotting number.

I.5. An inequality involving the number of positive circles.

Definition. Let D be a link diagram. Let $k(D)$ denote the number of Seifert circles of D. We call a Seifert circle *positive* (or *negative*) if it is only adjacent to positive (or negative) crossings. If D is positive, let $k_p(D) := k(D) - 1$, otherwise let $k_p(D)$ denote the number of positive Seifert circles of D. Let $k_n(D)$ be defined similarly for negative D and negative circles. If it does not lead to ambiguity, we just write k, k_p and k_n instead of $k(D), k_p(D)$ and $k_n(D)$.

Proposition I.25. Let D be a non-split diagram of writhe w of an oriented link L. Then

$$1 + w - k + 2k_n \leq s(L) \leq -1 + w + k - 2k_p$$

Proof. The upper bound is an easy corollary of the lower one and proposition I.23 (ii): note that the writhe of $\overline{D}$ is $-w$ and $k_n(\overline{D}) = k_p(D)$. Therefore we have

$$\begin{aligned} s(\overline{L}) &\geq 1 - w - k + 2k_p \Rightarrow \\ -s(\overline{L}) &\leq -1 + w + k - 2k_p \Rightarrow \\ s(L) - (s(L) + s(\overline{L})) &\leq -1 + w + k - 2k_p \Rightarrow \\ s(L) &\leq -1 + w + k - 2k_p. \end{aligned}$$

We prove the lower bound by constructing a representative of $\mathfrak{s}_o$ with the degree $w - k + 2k_n$. We start with the canonical representative; in (i), we show how to find – under certain conditions – another representative the degree of which is higher by 2. In (ii), we show (i) can be applied k_n times.

(i) Let c_- be a negative circle in s_o. Let $c_0 \in k(s_o)$ be a not necessarily negative circle adjacent to c_-. Let $\mathfrak{s} \in C_{s_o}$ color any negative circle but c_- and c_0 in an arbitrary way; and color c_-, c_0 and any non-negative circle the same way $\mathfrak{s}_o$ does. Then we say c_- is a *candidate circle of* $\mathfrak{s}$, c_0 its *helper circle*, and claim that the element $\mathfrak{s}' \in C_{s_o}$ that colors the candidate circle with $\mathbf{v}_+$ and every other circle in the same way $\mathfrak{s}$ does is homologous to $\mathfrak{s}'$.

Proof: Fix a common crossing of c_- and c_0. Let s be the state that resolves all crossings but the fixed one in the same way as s_o. Since this fixed crossing is negative, $h(s) = h(s_o) - 1$. Let c_1 be the merge of the circles c_- and c_0. Every circle of s but c_1 is a circle of s_o, too.

Let $\mathfrak{t} \in C_s$ color c_1 the same way $\mathfrak{s}$ colors the helper circle; and all other circles the same way $\mathfrak{s}$ does. What is the image of $\mathfrak{t}$ under $\partial_{h(s)-1}$? The fixed crossing is 0-resolved by s; passing to a 1-resolution splits c_1 in c_- and c_o, therefore $\partial_{s,s_o}(\mathfrak{t})$ colors every circle the same way $\mathfrak{s}$ does, but for the candidate circle: that circle is colored the way $\mathfrak{s}$ colors the helper circle. So $\partial_{s,s_o}(\mathfrak{t})$ colors the candidate circle in the opposite way $\mathfrak{s}$ does.

Any other 0-resolved crossing in s connects two circles that are colored oppositely by $\mathfrak{t}$. Therefore $\partial_{h(s)-1}(\mathfrak{t}) = \partial_{s,s_o}(\mathfrak{t})$. Let $\mathfrak{s}' = \mathfrak{s} - \partial_{h(s)-1}(\mathfrak{t})$: then $\mathfrak{s}'$ colors every circle the same way $\mathfrak{s}$ does, but for the candidate circle, which it colors $\pm\mathbf{v}_+$.

(ii) There is a sequence $(c_1, d_1), \dots (c_{k_n}, d_{k_n})$ of pairs of circles in $k(s_o)$, such that $\forall i : c_i$ is negative, shares a crossing with d_i, and $d_i \neq c_j$ for $j \leq i$.

Once (ii) has been proven, let $\mathfrak{s}_i \in C_{s_o}$ color c_j with $\mathbf{v}_+$ for $1 \leq j \leq i$, and every other circle the way $\mathfrak{s}_o$ does. Then c_{i+1} is a candidate circle of $\mathfrak{s}_i$ with helper circle d_{i+1}; so $[\mathfrak{s}_o] = [\mathfrak{s}_0] = [\mathfrak{s}_1] = \dots = [\mathfrak{s}_{k_n}]$ and $\deg_{C_{s_o}} \mathfrak{s}_{k_n} = w - k + 2k_n$.

Proof: Let Γ be the graph with vertices $k(s_o)$; and let two vertices be connected by an edge if and only if the two corresponding Seifert circles have a crossing in common. Let $T \subset \Gamma$ be a spanning tree: that is a tree with the same vertices as Γ the set of edges of which is a subset of the set of edges of Γ. We want to construct the above sequence inductively; start with an empty sequence. Let c be a leaf of T and c_0 its father. Prune c; and if c is negative, add (c, c_0) to the sequence. Proceed until only one vertex is left, in such a way that, if there is a non-negative circle, this last vertex is a non-negative circle. In this way, one arrives at a sequence $(c_1, d_1), \dots (c_{k_n}, d_{k_n})$ that satisfies $d_i \neq c_j$ for $j \leq i$.

□

Remark I.26. Let L be a link and D be a non-split diagram of L with the following property: only one Seifert circle of D is adjacent to positive as well as negative Seifert circles. Such a diagram is called *good*. Then the lower bound of the above proposition equals the upper; so the inequalities determine the Rasmussen invariant and we have $s(L) = w + k_n - k_p$.

Corollary I.27. Let D be a positive diagram of a link L, i.e. a diagram with positive crossings only. Then $s(L) = 1 + n - k$.

Similarly, if D is a negative diagram of L, $s(L) = -1 - n + k$.

Corollary I.28. If a link L has a good diagram, it satisfies $s(\overline{L}) = -s(L)$.

Remark I.29. The inequality is a refinement of the inequality $1 + w - k \leq s(L) \leq -1 + w + k$ that was proven by Shumakovitch [22].

It was proven by Kawamura [14] by constructing a cobordism to a positive link, the inequality I.20 then giving exactly the result; this has the advantage that the inequalities are proven for *every* knot invariant satisfying certain conditions[3]: the other prominent member of that set being the Ozsváth-Szabó invariant τ.

[3]That is, the invariant is required to be a homomorphism $\mathrm{Conc}(S^3) \to \mathbb{Z}$ that takes the value of twice the slice genus on positive knots and that has an absolute value smaller than twice the slice genus on any knot.

Remark I.30. Not all alternating knots have good diagrams; however, among small alternating knots, good diagrams are the rule: there are 1226 alternating knots of crossing number 12 or less; a computer calculation shows that for 1200 of them, the diagram given by the PD-notation from KnotInfo [9] is good.

For non-alternating knots with at most 12 crossings, it is 441 out of 1126.

I.6. **The relationship of the signature and s.** We make use of a statement obtained by Lee [18, proposition 3.3] using Goeritz matrices [12]:

Lemma I.31. Let D be a non-split, reduced, alternating diagram of a link L with n_+ positive crossings. It is possible to color the regions of D with black and white in a checkerboard fashion so that locally, every crossing looks like the crossing depicted on the right. Let o denote the number of black regions. Then $\sigma(L) = 1 + n_+ - o$.[4]

Remark I.32. Let D be an alternating diagram. A Seifert circle of D is called *separative* if there is at least one circle it contains and at least one it is contained in. Because D is alternating it has the following property: Every Seifert circle c of D shares only positive crossings with circles not contained in c, and only negative crossings with circles contained in c, or vice versa. So a circle is separative if and only if it is neither postive nor negative. Hence D is good if and only if there is at most one separative circle.

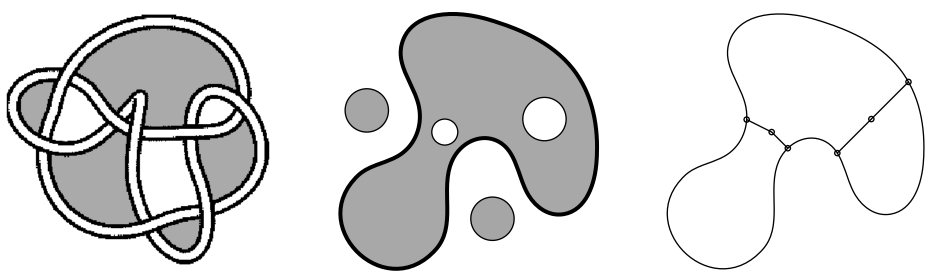


FIGURE 6. A diagram of 8_{12} colored like in lemma I.31 (graphic taken from Knot Atlas [5]); its colored Seifert resolution (separative circle drawn thicker) with $o = 4$; and one of the graphs mentioned in the proof of (i)

Proposition I.33. Let L be a non-split alternating link. Then $s(L) = \sigma(L)$.

Proof. Let D be a reduced alternating non-split diagram of L. Without loss of generality, let a crossing that connects two circle that are not contained in another one be positive. Let K_0 be the set of circles that are contained in an even number of other circles, and K_1 the set of those that are contained in an odd number. Then we prove that (i) $\sigma(L) = w + \#K_1 - \#K_0 + 1$ and (ii) $s(L) = w + \#K_1 - \#K_0 + 1$. Color D in the way of lemma I.31 above and let o be the number of black regions.

(i) Using Lee's lemma I.31, it suffices to prove $1+n_+-o = w+\#K_1-\#K_0+1 \Leftrightarrow o = \#K_0 - \#K_1 + n_-$. Consider a circle $c \in K_0$. We prove that the number o^c

[4] Lee follows a different convention regarding the sign of the signature than we do; so in Lee's paper, the formula actually appears as $\sigma(L) = o - n_+ - 1$.

of black regions inside of c equals $1 - k_1^c + n_-^c$. By summing over all $c \in K_0$, this implies $o = \#K_0 - \#K_1 + n_-$.

Consider the K_1-circles inside of c as vertices of a graph; let the edges between two K_1-circles c_1 and c_2 correspond to the crossings c_1 and c_2 have in common. Draw c as a circle containing these vertices. For every c_1 that shares a crossing with c, place a vertex on c and join it to c_1 with an edge. See 6 for an example of such a graph.

Let P be the number of negative crossings of c. Then the number of vertices of the graph is $P + k_1^c$; the number of edges is $P + n_-^c$; and the number of faces is o^c. Therefore, $P + k_1^c - P - n_-^c + o^c = 1 \Rightarrow o^c = 1 - k_1^c + n_-$.

(ii) By induction on the number of separative circles. If there is only one separative circle, D is good; furthermore, all circles in K_0 but one are positive and K_1 is the set of negative circles, so the claim is an application of I.26.

Now, choose a separative circle c. Let c have P positive crossings. Let D^i be the diagram consisting of c and the interior of c. Let K_0^i, K_1^i and w^i be set of Seifert circles of D^i contained in an even number of other circles, the set of circles contained in an odd number of other circles, and the writhe of D^i, respectively. Note that D^i has one separative circle less than D, because c is no longer separative in D. So by induction $s(L^i) = w^i + \#K_1^i - \#K_0^i + 1$, where L^i denotes the link represented by D^i.

Let D^e be the diagram consisting the exterior of c, but not c itself nor any crossing adjacent to c. Let K_0^e, K_1^e, w^e and L^e be defined similarly as above. Since D^e has one separative circle less than D, too, $s(L^e) = w^e + \#K_0^e - \#K_1^e + 1$ holds.

We have

$$\begin{aligned} w^i + w^e - N &= w \\ \#K_0^i + \#K_0^e &= \#K_0 \\ \#K_1^i + \#K_1^e &= \#K_1 \end{aligned}$$

There is a sequence of N fusion moves that connects D to $D^i \sqcup D^e$: just resolve all negative crossings adjacent to c. Therefore,

$$\begin{aligned} |s(L) - s(L^i \sqcup L^e)| &\leq N \Rightarrow \\ |s(L) - s(L^i) - s(L^e) + 1| &\leq N \Rightarrow \\ |s(L) - w^i - \#K_1^i + \#K_0^i - 1 - w^e - \#K_1^e + \#K_0^e - 1 + 1| &\leq N \Rightarrow \\ |s(L) - w - N - \#K_1 + \#K_0 - 1| &\leq N \Rightarrow \\ |s(L) - (w + \#K_1 - \#K_0 + 1) - N| &\leq N \Rightarrow \\ s(L) - (w + \#K_1 - \#K_0 + 1) &\geq 0 \Rightarrow \\ w + \#K_1 - \#K_0 + 1 &\leq s(L) \end{aligned}$$

Apply this to $\overline{D}$; since now a crossing between circles that are not contained in another circle are egative instead of positive, we have to change the diagram slightly: pick a circle in K_0 that is not contained in another one. Then one can switch the interior and exterior of this circle, obtaining a diagram that represents the same link. Under this, $(K_0, K_1) \mapsto (K_1 + 1, K_0 - 1)$. Therefore

$$\begin{aligned} -w - \#K_1 + \#K_0 - 1 &\leq s(\overline{L}) \\ w + \#K_1 - \#K_0 + 1 &\geq s(L) \end{aligned}$$

The claim follows. □

Corollary I.34. If L is an alternating link with C split components, then $s(L) = \sigma(L) - C + 1$.

Remark I.35. If L is a link, $(\sigma(L) - s(L))/2 \leq \operatorname{alt}(L) \leq u(L)$, where $u(L)$ is the unknotting number number of L, and $\operatorname{alt}(L)$ is the *alternation number*: the minimum number of crossing changes needed to change a diagram of L to the diagram of an alternating link, the minimum taken over all diagrams of L.

II. Arborescent links

II.1. **Definition and basic properties of arborescent links.** This subsection follows Baader [1]. We start with a list of familiar definitions from graph theory:

List of definitions. A *G-weighted graph* Γ is a tuple (V, E, γ): V is the finite set of *vertices*, $\gamma : V \to G$ the *weight function*, and $E \subset \{\{v, w\} \mid v, w \in V, v \neq w\}$ the set of *edges*. A vertex v and an edge e are called *adjacent* if $v \in e$; two vertices v, w are called *adjacent* if $\{v, w\} \in E$. The set of edges adjacent to a given vertex v is denoted by E_v. To *prune* a subset $V' \subset V$ means to delete the vertices in V' and all edges they are adjacent to. A *path* in Γ is a non-empty sequence $v_1, \ldots v_n$ of distinct vertices so that $\forall i \in \{1, \ldots n-1\} : \{v_i, v_{i+1}\} \in E$. A path is said to *connect* v_1 *to* v_n. A graph is called a *forest* if for any pair of distinct vertices v, w there is at most one path from v to w; it is called a *tree* if for any pair there is one and only one. A *leaf* of a forest is a vertex adjacent to only one edge. The only vertex a leaf is adjacent to is called its *father*. Two G-weighted graphs $\Gamma_1 = (V_1, E_1, \gamma_1)$ and $\Gamma_2 = (V_1, E_1, \gamma_1)$ are considered the same if there is a *G-weighted graph isomorphism* $f : \Gamma_1 \to \Gamma_2$, i.e. a bijection $f : V_1 \to V_2$ such that $\{v, w\} \in E_1 \Leftrightarrow \{f(v), f(w)\} \in E_2$ and $\gamma_1 = \gamma_2 \circ f$.

A graph can be seen as a topological space: let $\preccurlyeq$ be a linear order on V. For each edge e let I_e be a unit interval. Let $e = \{v, w\} \in E$ be an edge and $v \preccurlyeq w$. Then we set $I_{e,v} = 0 \in I_e$ and $I_{e,w} = 1 \in I_e$. Consider the space $\bigsqcup_{e \in E} I_e / \sim$ where $I_{e,v} \sim I_{f,v}$ for $e, f \in E, v \in V$. This is the topological space corresponding to the graph. An *embedding* of a graph is an embedding of its topological space in $\mathbb{R}^2$. Let two embedded, G-weighted graphs $i_1 : \Gamma_1 \to \mathbb{R}^2$ and $i_2 : \Gamma_2 \to \mathbb{R}^2$ be given, and denote their topological spaces by X_1 and X_2. A graph isomorphism $f : \Gamma_1 \to \Gamma_2$ induces a homeomorphism $\widetilde{f} : X_1 \to X_2$. The embedded graphs Γ_1 and Γ_2 are considered the same if there is a G-weighted graph automorphism f such that $i_2 \circ \widetilde{f}$ and i_1 are isotopic.

An embedding of a graph induces a symmetric relation on each E_v that conveys if two edges are next to each other in the embedding.

Every forest is embeddable.

Definition. To a $\mathbb{Z}$-weighted, embedded tree $\Gamma = (V, E, \gamma)$ one associates an embedding of a surface in S^3 by plumbing twisted bands along the tree: first, place a $\gamma(v)$-twisted band (see Figure 7) at every vertex v. Then, for each edge $\{v, w\} \in E$, plumb together the band at v with the one at w as in figure 8, in the order given by the embedding: i.e., the plumbing sites of the bands at w and w' are next to each other if and only if the edges $\{v, w\}$ and $\{v, w'\}$ are next to each other in the embedding of Γ.

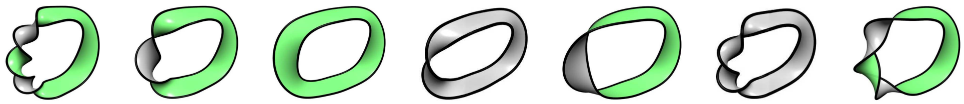


FIGURE 7. The −4-, −2-, 0- (cylinder), 1- (Möbius strip), 2- (Hopf band), 3-, and 4-twisted band. In the orientable case the two sides of the band are colored differently. In the making of this figure SeifertView [24] was used.

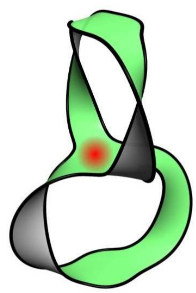


FIGURE 8. One 2- and one (−2)-twisted bands plumbed together forming a Seifert surface of 4_1, the plumbing site colored red. In the making of this figure SeifertView [24] was used.

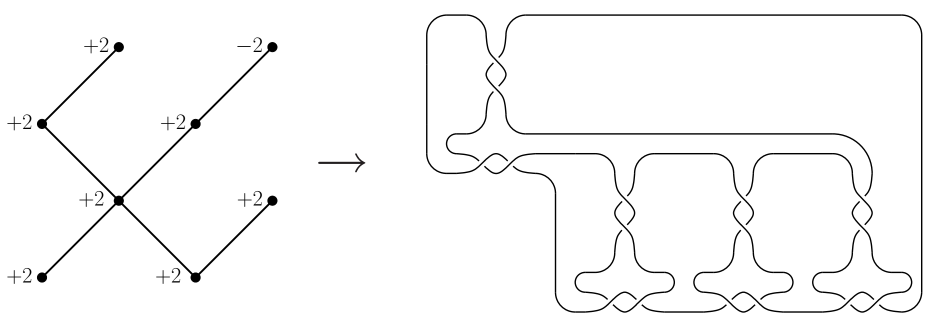


FIGURE 9. From the weighted, embedded tree to the diagram of the link bordering the surface.

This plumbing operation is a special case of the *Murasugi sum*; we rely on figure 8 and do not give an exact definition of plumbing. Such a definition can be found in Kawauchi's book [15, chaper 4.2].

The border of that surface is a link called the *link associated to* Γ.

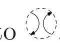
Remark II.1. Notice that two oriented surfaces yield an oriented surface if plumbed together. A n-twisted band is orientable if and only if $n \in 2\mathbb{Z}$. Thus, the surface associated to a $2\mathbb{Z}$-weighted tree is orientable and hence constitutes a Seifert surface for the link associated to the tree. We refer to this Seifert surface as the *canonical Seifert surface* of L. The existence of such a surface proves to be highly fruitful.

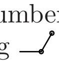
In the remainder of section II, let $\Gamma = (V, E, \gamma)$ with $V = (v_1, \ldots v_n)$ be a $2\mathbb{Z}$-weighted tree and L the link associated to Γ with an arbitrary embedding.

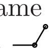
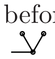
Lemma II.2. The number $c(\Gamma)$ of link components of L can be calculated recursively:

(i) If Γ has a single vertex, $c(\Gamma) = 2$.
(ii) If Γ has just two vertices, $c(\Gamma) = 1$.
(iii) If there is a leaf the father of which is adjacent only to one vertex aside from the leaf, one can prune the leaf and his father: the resulting tree Γ' satisfies $c(\Gamma') = c(\Gamma)$.
(iv) If there are two leafs with the same father, one can prune one of them: the resulting tree Γ' satisfies $c(\Gamma') = c(\Gamma) - 1$.

Proof. (i) and (ii) can be easily checked.

Plumbing is a local action and takes a place in a tangle of the type . If one plumbs a band b only once with a band b', the type of tangle is changed to .

If one plumbs the band b' with another band, the type of tangle is changed back to  and no additional circles emerge in the tangle. So the number of link components of the link associated to a tree is not changed by adjoining  to some vertex. This proves (iii).

On the other hand, if one plumbs b with another band, the type of the resulting tangle is ; that is the same as before, but an additional circle comes up in the tangle. Thus passing from  to  increases the number of link components of the link associated to the tree by 1. This proves (iv).

An easy induction argument shows that by applying (i)–(iv), c can be calculated for all trees: Apply (iii) and (iv) until one reaches (i) or (ii). □

II.2. **The signature of arborescent links.** The calculation of the signature of an arborescent link will follow the direct way: find a Seifert matrix of the link and manipulate that matrix to obtain its signature.

Definition. The *matrix associated to* Γ is denoted by A_Γ and defined as

$$(A_\Gamma)_{ij} = \begin{cases} w(v_i) & i = j \\ 1 & \{v_i, v_j\} \in E \\ 0 & \{v_i, v_j\} \notin E, \end{cases}$$

where we allow Γ to be a forest.

Lemma II.3. There is a Seifert matrix S of L so that $A_\Gamma = S + S^\top$.

Proof. Consider the natural Seifert surface of L (see remark II.1). As a basis of the first homology group of that surface we choose the set of $(f_v)_{v\in V}$, where f_v "goes around" the twisted band associated to the edge v one time.

It should be clear that $\mathrm{lk}(f_v, f_v^+) = w(v)/2$. The curves f_v and f_w^+ may only be linked if the band at v and the band at w are plumbed, i.e if and only if $\{v, w\} \in E$. It is possible to choose the orientations of the f_v so that $\mathrm{lk}(f_v, f_w^+) + \mathrm{lk}(f_w, f_v^+) = 1$.

Let S be the Seifert surface corresponding to that choice of basis. Then, clearly, $A_\Gamma = S + S^\top$. □

The following proposition gives an easy algorithm operating on Γ that computes the signature of A_Γ and thereby of L.

Proposition II.4. (i) If $E = \varnothing$, then $\sigma(A_\Gamma) = \sum_{v\in V} \mathrm{sig}(w(v))$.

Otherwise, let $v \in V$ be a leaf. Denote its father by $w \in V$.

(i) If $\gamma(v) \neq 0$, then $\sigma(A_\Gamma) = \sigma(A_{\Gamma'})$, where Γ' is obtained from Γ by removing the edge $\{v, w\}$ and changing the weight of w to $\gamma(w) - 1/\gamma(v)$.

(ii) If $\gamma(v) = 0$, then $\sigma(A_\Gamma) = \sigma(A_{\Gamma'})$, where Γ' is obtained from Γ by removing the two vertices v, w and all edges adjacent to them.

Proof. If $E = \varnothing$, A_Γ is a diagonal matrix with entries $\gamma(v_i)$; this proves (i).

While proving (ii) and (iii), for the sake of better legibility reindex the vertices so that $v = v_1$ and $w = v_2$. Then A_Γ has the form

$$\begin{pmatrix} \gamma(v) & 1 & & & \\ 1 & \gamma(w) & (A_\Gamma)_{23} & \cdots & (A_\Gamma)_{2n} \\ & (A_\Gamma)_{32} & & & \\ & \vdots & & \boxed{B} & \\ & (A_\Gamma)_{n2} & & & \end{pmatrix}$$

where left-out entries are understood to be zero and B is the matrix associated to the tree obtained from Γ by pruning v and w.

The signature of A_Γ is invariant under $A_\Gamma \mapsto S A_\Gamma S^\top$ where $S \in \mathrm{GL}_n(\mathbb{Q})$. In case of (ii), let

$$S = \begin{pmatrix} 1 & & & \\ -1/\gamma(v) & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

That results in

$$S A_\Gamma S^\top = \begin{pmatrix} \gamma(v) & & & & \\ & \gamma(w) - 1/\gamma(v) & (A_\Gamma)_{23} & \cdots & (A_\Gamma)_{2n} \\ & (A_\Gamma)_{32} & & & \\ & \vdots & & \boxed{B} & \\ & (A_\Gamma)_{n2} & & & \end{pmatrix}$$

which is the matrix associated to the tree obtained from Γ as described in the statement of (ii).

In case of (iii), let

$$S = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ -(A_\Gamma)_{23} & & 1 & & \\ \vdots & & & \ddots & \\ -(A_\Gamma)_{2n} & & & & 1 \end{pmatrix}.$$

That results in

$$S A_\Gamma S^\top = \begin{pmatrix} 0 & 1 & \\ 1 & \gamma(w) & \\ & & \boxed{B} \end{pmatrix}.$$

That matrix has the same signature as B, because

$$\det\begin{pmatrix}0 & 1\\ 1 & \gamma(w)\end{pmatrix} = -1 \Rightarrow \sigma\begin{pmatrix}0 & 1\\ 1 & \gamma(w)\end{pmatrix} = 0.$$

□

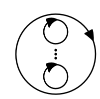
Remark II.5. Note that signature and Rasmussen invariant may agree on non-alternating arborescent links, e.g. on 8_{19}, or may as well disagree, e.g. on 10_{139} or 10_{152}.

II.3. **The Rasmussen invariant of arborescent links.**

Lemma II.6. Let D be the canonical diagram of L with $\sum_{v\in V}|\gamma(v)|$ crossings (cf figure 9) and writhe $\sum_{v\in V}\gamma(v)$. Then the Seifert resolution of D has

- $1+\sum\limits_{v\in V}(|\gamma(v)|-1)$ many Seifert circles.
- $\sum\limits_{\substack{v\in V\\ \gamma(v)>0}}(\gamma(v)-1)$ many positive circles.
- $\sum\limits_{\substack{v\in V\\ \gamma(v)<0}}(-\gamma(v)-1)$ many negative circles.

Proof.
By induction on $\#V$. For $\#V=1$, a Seifert diagram is shown on the right; the claim holds for this diagram. We call the outer circle the *big circle*.



Now let the claim be true for Γ, and modify Γ by adding one edge and one vertex v. This has the following effect on the Seifert diagram of the link associated to Γ:

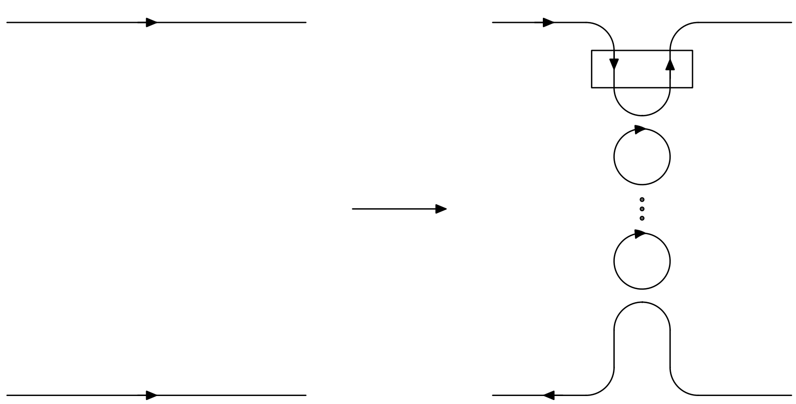


FIGURE 10. Effect of plumbing on the Seifert diagram.

First, we prove inductively that the two arcs depicted in the left diagram always belong to the big circle. This is true if v is the second vertex of Γ. If one adds a vertex v' by adjoining it to v, the diagram is changed inside the rectangle drawn in the right diagram: again both arcs belong to the big circle. The same is true if the vertex v' is added to the father of v.

There are $\gamma(v)-1$ new Seifert circles, all of which are positive if $\gamma(v)$ is positive, and are negative if $\gamma(v)$ is. There are $\gamma(v)$ new crossings with the same sign as $\gamma(v)$. The only old Seifert circle that gained new crossings is the big circle. Hence all other circles that were positive or negative before are so still. Adding these changes to the induction hypothesis proves the claim. □

Proposition II.7. Let Γ be a $2\mathbb{Z}$-weighted tree and L the link associated to Γ embedded arbitrarily. Then

$$s(L) = \sum_{v\in V} \operatorname{sig}(\gamma(v)).$$

Proof. Note that by the previous lemma, the canonical diagram D of L is good; so following remark I.26 we have

$$\begin{aligned} s(L) &= w(D) + k_n(D) - k_p(D) \\ &= \sum_{v\in V}\gamma(v) + \sum_{\substack{v\in V\\ \gamma(v)<0}}(-\gamma(v)-1) - \sum_{\substack{v\in V\\ \gamma(v)>0}}(\gamma(v)-1) \\ &= \sum_{v\in V}\operatorname{sig}(\gamma(v)). \end{aligned}$$

□

Alternative proof. Let $V_+ = \{v \in V \mid \gamma(v) > 0\}$ and $V_- = \{v \in V \mid \gamma(v) < 0\}$. In the canonical diagram of L, one can apply one fusion move and $|n|$ Reidemeister I-moves to the part of the diagram that comes from an n-twisted band in order to eliminate that band. If one does this to all negative bands, one obtains a positive diagram D_+ of an arborescent link L_+ coming from the complete subgraph containing all positive vertices of Γ. The link L_+ is positive, therefore $s(L_+) = 1 + c(D_+) - k(D_+) = \#V_+$ (see corollary I.27).Thus $|s(L) - \#V_+| \leq \#V_-$.

Similarly, one can eliminate all positive bands at the cost of $\#V_+$ fusion moves to arrive at a negative link with Rasmussen invariant $-\#V_-$, and therefore $|s(L) - \#V_-| \leq \#V_+$.

Together, these two inequalities imply that $s(L) = \#V_+ - \#V_-$. □

Remark II.8. The Rasmussen invariant does not always provide a better estimate for the four-genus than the signature: for the knot K coming from the tree in figure 11, we have $s(K) = 0$, but $\sigma(K) = 2$.

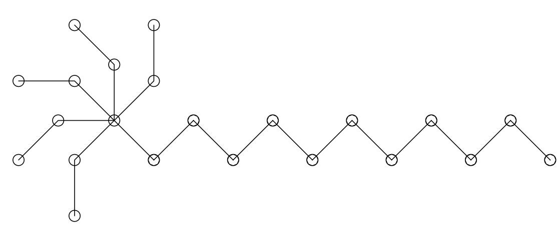


FIGURE 11. A scorpion with positive body, five positive legs and one negative tail 11 vertices long (vertices marked with a circle have weight -2, vertices marked with a dot weight $+2$.)

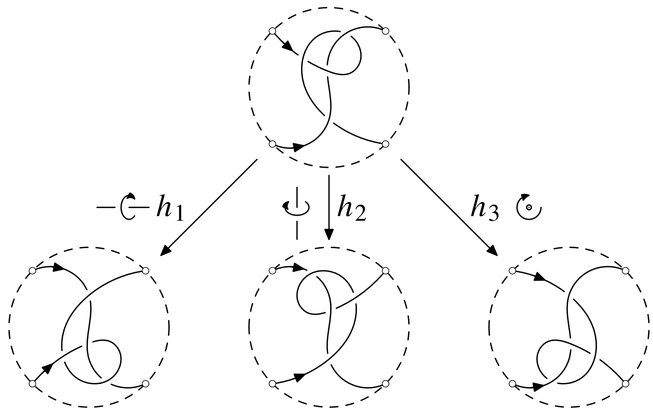
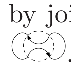


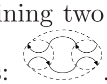
FIGURE 12. The three half-turns.

III. MUTANT LINKS

III.1. **Definition of mutant links.** Our definitions follow Wehrli [27], who in turn follows Murakami [19].

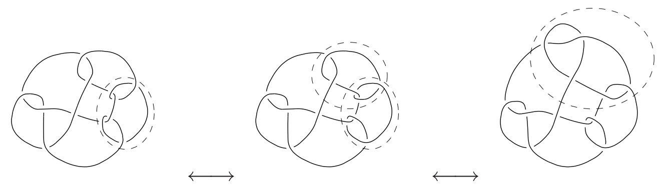
List of definitions. An oriented 2-tangle is a smooth embedding $S^1 \sqcup \ldots \sqcup S^1 \sqcup I \sqcup I \to D^3$ of two oriented arcs and an arbitrary finite number of oriented circles into a three-ball such that the intersection with the boundary of the ball consists of the endpoints of the arcs.[5] A tangle diagram resembles a knot diagram; the ball is drawn as a dashed circle. From the diagrammatic viewpoint, there are three different types of tangles: ,  and . Three-dimensionally seen, however, they are equivalent, so in the following definitions we always assume our tangle to be of the first type.

The *closure* of a tangle is the link obtained by joining the four endpoints such that the two arcs of the tangle form one circle: .

The *sum* of two tangles is the tangle obtained by joining two endpoints of one tangle to two endpoints of the other tangle by two arcs: . Note that there are two different possibilities to sum two tangles.

Now assume we allow the reversal of the orientation of one of the tangles before adding. Then there are four possibilities to compose two tangles. Let L be an oriented link that is the closure of a sum of two tangles T_1 and T_2. Add T_1 and T_2 in one of the three other possible ways, reversing the orientation of T_2 if necessary; this yields a link L'. Passing from L to L' is called *elementary mutation*. Links that are connected by a sequence of elementary mutations are called *mutant links*. Elementary mutation can be thought of as first cutting the two tangles, then applying one of the three half-turns depicted in figure 12, and finally joining the tangles again. Note that two of the half-turns (h_2 and h_3) make it necessary to reverse the orientation of one of the tangles.

[5]Or equivalently: a 2-tangle is the intersection of a three-ball with a link such that the boundary of the three-ball intersects the link in four points.



$12n_{56}$ $\longleftrightarrow$ $12n_{57}$ $\longleftrightarrow$ $\overline{12n_{221}}$

FIGURE 13. Example of a triple of mutant knots. The knot diagrams are modified diagrams from Knotscape [13].

Remark III.1. Examples for mutant links are:

- Decompose a knot as sum of two tangles in such a way that mutation gives the same isotopy class of unoriented knot. Then, applying h_2 or h_3 reverses the orientation of the knot. So, 8_{17} and its reverse constitute the pair of distinct mutant knots with the lowest crossing number.
- The most famous pair of mutant knots is the Kinoshita-Terasaka knot $11n_{42}$ and the Conway knot $11n_{34}$.
- A complete list of mutant knots with up to 15 crossings, disregarding orientation, can be found on Alexander Stoimenow's homepage [23].
- Yvonne Gerber proved in her dissertation [11] that two different embedded trees that have weights equal to $+2$ produce two different knots. This fact allows to construct easily arbitrarily large classes of arbitrarily large mutant knots. For arborescent links coming from trees with arbitrary even weights, the analogous statement seems not to have been proven. An example is the knot depicted in figure 9 ($12n_{220}$), which is different from the knot produced by switching two branches of the vertex of degree 4 of the given tree: that is the knot $12n_{59}$.
- The pretzel links $P(a_1, \dots a_n)$ and $P(\sigma(a_1), \dots \sigma(a_n))$ are mutant for an arbitrary permutation $\sigma \in S(n)$; for many choices of σ, however, the two pretzel links are not the same. This example is just a special case of the previous, since the link $P(a_1, \dots a_n)$ is arborescent: the corresponding tree has one vertex v with weight 0, and n vertices with weights a_i that all are adjacent to v.

III.2. **Behaviour of the Rasmussen invariant under mutation.** An elementary mutation is called *component-preserving* if the two arcs of the tangle that is being half-turned belong to the same link component. A mutation is called component-preserving if it is the composition of component-preserving elementary mutations.

Khovanov homology is not invariant under general mutation; there is a simple example of mutant links with different Khovanov homology (published by Wehrli [25]), namely $3_1 \sqcup 3_1$ and $(3_1 \# 3_1) \sqcup 0_1$.

However, there is hope that the Khovanov homology might be invariant under component-preserving mutation of links; there is an "almost-proof" by Bar-Natan [4], which has been successfully completed to a proof by Wehrli for Khovanov homology over $\mathbb{F}_2$ (see [26]). Recently, a proof for mutation invariance of odd Khovanov homology was published by Bloom [8].

Furthermore, Khovanov homology is conjectured to determine the Rasmussen invariant of knots [2], a fact already proven by Lee for alternating knots [17]. Putting the conjectures together, s is conjectured to be invariant under mutation of knots; there is no pair of mutant links – even dropping the condition that components are preserved – with different Rasmussen invariant known, either, but since the Rasmussen invariant has not been tabulated for small links, that need not be too strong evidence.

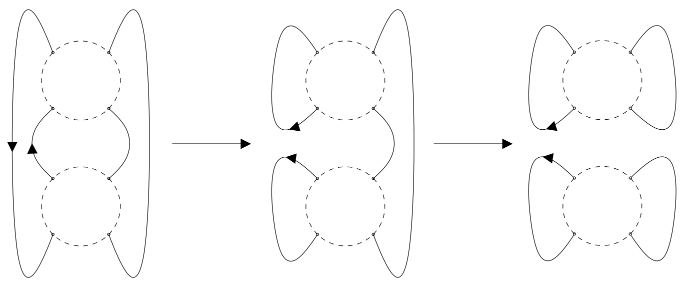


FIGURE 14. Two fusions decomposing a sum of two tangles.

Proposition III.2. Let L_1 and L_2 be a pair of mutant links, then $|s(L_1)-s(L_2)| \leq 4m$, where m is the minimal number of half-turns necessary to carry out the mutation.

Proof. First, apply two fusion moves as shown in figure 14. This yields the disjoint sum of two links L_1 and L_2. Now, rotate L_2 according to the kind of half-turn given, and reverse its orientation if necessary. This does not change the Rasmussen invariant. Finally, apply again two fusion moves, retracing the steps shown in figure 14. During this whole process, four fusion moves were carried out, so the Rasmussen invariant changes by at most 4 (see lemma I.20). Repeat for every half-turn. □

Remark III.3. If $s(L_1\#L_2) = s(L_1)+s(L_2)$ holds for arbitrary L_1 and L_2 (see I.23, (iii)), the inequality of the above proposition can be improved to $|s(L_1) - s(L_2)| \leq 2m$.

Conclusion: Results and open problems

In the first section, a little improvement could be made regarding the estimate of $s(L)+s(\overline{L})$: we proved $s(L)+s(\overline{L}) \leq 0$, and only $s(L)+s(\overline{L}) \leq 2$ was known before. Apart from that, the first section does not contain any new results. However, alternative proofs for Kawamura's inequality and the equality of the signature and the Rasmussen invariant on alternating non-split links were given. Furthermore, we defined the Rasmusssen invariant and proved almost all its basic properties without using spectral sequences: the only part that is still missing is the proof of Reidemeister invariance of the homology of the Khovanov-Lee complex.

Generally, the Rasmussen invariant of links does not seem to draw much attention. It is not yet well understood how $s(L) + s(\overline{L})$ behaves, e.g. if it is possible to further characterize links satisfying $s(L) + s(\overline{L}) = 0$. To show $s(L_1 \# L_2) = s(L_1) + s(L_2)$ is an open problem as well. The relationship between the Rasmussen invariant and Khovanov homology on links is not yet well-known, either. Finally, tabulating the Rasmussen invariant of small links – as has been done for small knots – would be helpful for further exploration.

In the second section, the signature and Rasmussen invariant of trees with even weights could be computed; it is not clear yet if this might help in determining their slice genus. Furthermore, the Rasmussen invariant or signature of arbitrary weighted trees could be interesting; as well as the effect on the Rasmussen invariant of plumbing arbitrary links with twisted bands or even plumbing two arbitrary links.

In the third section, an upper bound for the difference of the Rasmussen invariant of mutant knots was found; an attempt to actually prove mutation invariance could be made by imitating Bar-Natan's "almost-proof" ([4]), using Bar-Natan's local approach to Khovanov-Lee homology ([6]). Or, since odd Khovanov homology is mutation invariant (see Bloom [8]), it would also suffice to prove that odd Khovanov homology determines the Rasmussen invariant.

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